Frostman Lemma.

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Let us now formulate an inverse to Mass Distribution principle.

Thm (Frostman Lemma). Let h be a gauge function, $H_h(K)>0$ for some $K\subset\mathbb{R}^d$. Then $\exists \mu: \mu(K)\geq H_h(K)$, $\mu(B(x,r))< C_d h(r)$, for some C_d depending only on d.

Proof. $\forall$ binary cubes. Note that if we consider any covering of K by the binary cubes of sizes 2^{-n_i}, then $\sum h(2^{-n_i})\geq \frac{H_h(K)}{C_d}$ (C_d is a constant, since each 2^{-l_i} cube can be covered by at most $(\sqrt{d})^d$ cubes of diameter 2^{-l_i}). To make things non-intersecting, let us, as usual, make our cubes semi-open. WLOG, by scaling, can assume $K\subset Q_0$ - 1-diadic cube.

Fix $n>0$. Let $L_n:=\{Q: Q - 2^{-n}\text{ cube}: Q\cap K\neq\emptyset\}$

Let us define μ_n on $\bigcup_{Q\in L^n} Q\supset K$ the following way.

μ_n^1 would have constant density on each $Q\in L_n$, with $\mu_n^1(Q)=h(2^{-n})$.

Then $\forall\, Q\in L_{n-1}$. If $\mu_n^1(Q)\leq h(2^{-n+1})$, then let $\mu_n^2=\mu_n^1$ in Q. If not, let, for $E\subset Q$, $\mu_n^2(E)=\frac{h(2^{-n+1})}{\mu_n^1(Q)}\mu_n^1(E)$ - rescale μ^1.

keep doing the rescaling till we constructed $\mu_n:=\mu_n^n$. μ_n is supported on $\overline{\bigcup_{Q\in L_n} Q}$.

Call a 2^{-k}-cube Q good if $\mu_n(Q)=h(2^{-k})$. Good cubes cover $\bigcup_{Q\in L_n} Q$ (since for any x we can look at the last time we rescaled μ when we replaced the measure). So we can select non-intersecting cover by them, N_n, and

$$\mu_n\Big(\bigcup_{Q\in L_n}\Big)=\mu\Big(\bigcup_{Q\in N_n} Q\Big)=\sum_{Q\in N_n} h\big(2^{-k(Q)}\big)\geq\frac{H_h(K)}{C_d}; \text{ and } \mu_n\Big(\bigcup_{Q\in L_n} Q\Big)\leq h(1)$$

By construction, $\forall$ any 2^{-m} cube Q with $m\leq n$, we have $\mu_n(Q)\leq h(2^{-m})$.

By Banach-Alaoglu Thm, $\exists$ subsequence n_j: such that $\mu_{n_j}\to\mu$ weak*, i.e. $\forall$ continuous φ, $\int\varphi\, d\mu_{n_j}\to\int\varphi\, d\mu$.

Then μ is supported on $K = \bigcap_n \bigcup_{Q \in C_n} Q$.

$\mu(K) \leq \mu(Q_0) = \lim_{n\to\infty} \int 1 \, d\mu_n = \lim_{n\to\infty} \mu_n(Q_0) \geq \frac{H_h(K)}{e_d}$.

Fix now a 2^{-m} cube Q, and let φ be continuous, equal to 1 on Q, and $= 0$ outside of 2^{-m} nbhd of Q. Then note that $\int \varphi \, d\mu_n \leq 3^d h(2^{-m})$ (since 2^{-m} nbhd of Q can be covered by 3^d 2^{-m}-cubes). So
$\mu(Q) \leq \int \varphi \, d\mu = \lim_{n\to\infty} \int \varphi \, d\mu_n \leq 3^d h(2^{-m})$.

Now any $B(x,r)$ can be covered by some C_d diadic cubes of size $< r$. Thus $\mu(B(x,r)) \leq 3^d \cdot C_d \, h(r)$.
By rescaling μ by C_d, we get the required measure ∎